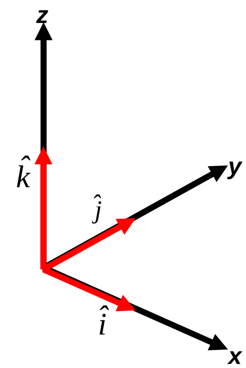
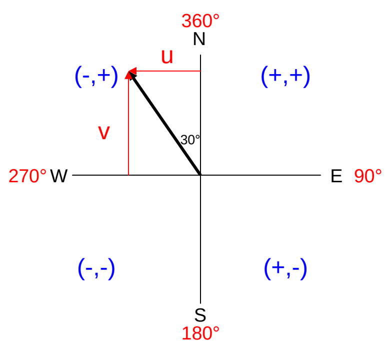
Vectors and Vector Analysis

- Scalar – Representation of a quantity using only a number (e.g., temperature).
- Vector – Representation of a quantity that has both magnitude and direction (e.g., wind velocity).
- Unit Vectors – Vectors with unit length (i.e., magnitude = 1) that are parallel to the coordinate axes.



Representing Wind As a Vector



$$\left|\vec{V}\right| = 21\,knots$$

$$\left|u\right| = 21\sin 30^\circ\,knots$$

$$\left|v\right| = 21\cos 30^\circ\,knots$$

$$u = -10.5\,knots$$

$$v = 18.2\,knots$$

$$\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$$

X

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Magnitude of a Vector

$$\left|\vec{A}\right| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

Example:

$$\left|\vec{V}\right| = \sqrt{u^2 + v^2 + w^2}$$

Basic Vector Operations

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

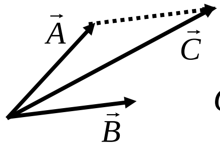
Multiplication of a vector by a scalar:

$$c\vec{A} = \left(cA_x\right)\hat{i} + \left(cA_y\right)\hat{j} + \left(cA_z\right)\hat{k}$$

(change in magnitude without change in direction)

Begin with: $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$



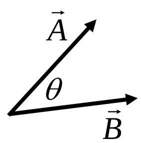
Vector Addition:

$$\vec{C} = \vec{A} + \vec{B} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} + (A_z + B_z)\hat{k}$$

Vector Subtraction:

$$\vec{C} = \vec{A} - \vec{B} = (A_x - B_x)\hat{i} + (A_y - B_y)\hat{j} + (A_z - B_z)\hat{k}$$

Scalar or Dot Product:



$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

$$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\, cos\,\theta$$

Note: If $\vec{A} \perp \vec{B}$ then $\vec{A}\cdot\vec{B} = 0$

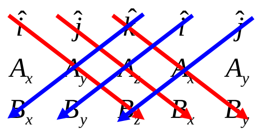
$$\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$$

$\vec{A}\cdot\vec{B}$ is a scalar quantity

Vector or Cross Product:

$$\vec{A}\times\vec{B}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}=\begin{vmatrix}A_y & A_z\\ B_y & B_z\end{vmatrix}\hat{i}-\begin{vmatrix}A_x & A_z\\ B_x & B_z\end{vmatrix}\hat{j}+\begin{vmatrix}A_x & A_y\\ B_x & B_y\end{vmatrix}\hat{k}$$

$$\vec{A}\times\vec{B}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}=\left(A_yB_z-A_zB_y\right)\hat{i}-\left(A_xB_z-A_zB_x\right)\hat{j}+\left(A_xB_y-A_yB_x\right)\hat{k}$$



$$\rightarrow\ \left(A_yB_z-A_zB_y\right)\hat{i}+\left(A_zB_x-A_xB_z\right)\hat{j}+\left(A_xB_y-A_yB_x\right)\hat{k}$$

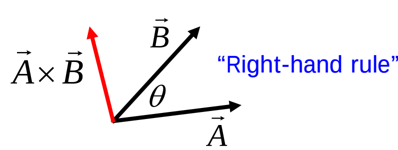
Vector or Cross Product:

$$\vec{A}\times\vec{B}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}=\left(A_yB_z-A_zB_y\right)\hat{i}-\left(A_xB_z-A_zB_x\right)\hat{j}+\left(A_xB_y-A_yB_x\right)\hat{k}$$

Note: $\left|\vec{A}\times\vec{B}\right|=\left|\vec{A}\right|\left|\vec{B}\right|\sin\theta$

$\vec{A}\times\vec{B}=0\rightarrow\vec{A}\parallel\vec{B}$

$\vec{A}\times\vec{B}=-\vec{B}\times\vec{A}$



$\vec{A}\times\vec{B}$ is a vector quantity

Triple Products (Caution!)

$$\vec{A}\times\left(\vec{B}\times\vec{C}\right)\neq\left(\vec{A}\times\vec{B}\right)\times\vec{C}$$

$$\left(\vec{A}\cdot\vec{B}\right)\vec{C}\neq\vec{A}\left(\vec{B}\cdot\vec{C}\right)$$

$\vec{A}\times\vec{B}\times\vec{C}$ is undefined.

$\left(\vec{A}\cdot\vec{B}\right)\times\vec{C}$ is undefined.

$\left(\vec{A}\cdot\vec{B}\right)\cdot\vec{C}$ is undefined.

Uses of the del (or gradient) operator

If $f = f(x,y,z,t)$ is a scalar function, then

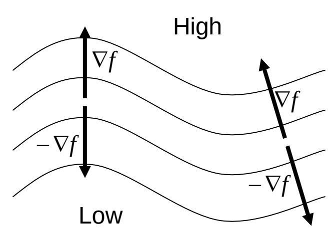
$$\nabla_3 f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

indicates gradient is computed in 3 dimensions

horizontal gradient $\nabla_2 f$

vertical gradient

∇f is a vector that points in the direction of most rapid increase of f at a given point, and ∇f points from low to high values of f.



Euler's relation (expansion of total derivative):

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z}$$

$$u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z} = \vec{V}\cdot\nabla f$$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \vec{V}\cdot\nabla f$$

Divergence of a Vector

$$div\ \vec{A} = \nabla\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

(Divergence is a scalar quantity.)

Example (divergence of the wind):

$$\nabla\cdot\vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

Curl of a Vector

$$curl\ \vec{A} = \nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{k}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} & \hat{i} & \hat{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ A_x & A_y & A_z & A_x & A_y \end{vmatrix}$$

(Curl is a vector quantity.)

Laplacian of a Scalar

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

(The Laplacian is a scalar quantity.)